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Efficient Solution of Shifted Quasiseparable Systems and Applications

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Abstract

We propose an efficient algorithm for the solution of shifted quasiseparable systems, which exploits the invariance of the quasiseparable structure under diagonal shifting and inversion. This algorithm is applied to compute various functions of matrices, and to solve a class of linear matrix equations. Numerical experiments show the effectiveness of our approach.

Key words: quasiseparable matrices; shifted linear system; QR factorization; matrix function; matrix equation.
MSC 2000: 65F05

1 Introduction

In this work we propose a novel method for computing the solution of shifted quasiseparable systems of the form

$$(A + \sigma_i I_N)\boldsymbol{x}_i = \boldsymbol{y}, \quad i = 1, \dots, \ell, \tag{1}$$

and of more general parameter dependent linear matrix equations with quasiseparable representations. Our approach also has a noticeable potential for effectively solving some large-scale algebraic problems that reduce to evaluating the action of a quasiseparable matrix function to a vector.

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Quasiseparable matrices find their application in several branches of applied mathematics and engineering. For instance, quasiseparable structure often arises in the discretization of continuous operators, due to the local properties of the discretization schemes and/or to the decay properties of the operator or of its finite approximations. As a consequence, in the last decade there has been considerable interest in the development of fast algorithms for working with quasiseparable matrices [3, 4, 9, 10].

2 Main Algorithm

It is well known that several operations with quasiseparable matrices can be performed in linear time with respect to matrix size. In particular, the QR factorization algorithm presented in [2] computes in linear time a QR decomposition of a quasiseparable matrix $A \in \mathbb{C}^{N \times N}$ of the form $A = V \cdot U \cdot R$, where R is upper triangular, whereas U and V are banded unitary matrices and – this is the crucial point – V only depends on the generators of the strictly lower triangular part of A. This implies that any shifted matrix $A + \sigma I_N$, $\sigma \in \mathbb{C}$, can also be factored as $A + \sigma I_N = V \cdot U_\sigma \cdot R_\sigma$ for suitable U_σ and R_σ.

Relying upon this fact, we design and implement an efficient algorithm [1] for solving a sequence of shifted quasiseparable linear systems. Its arithmetic complexity is linear with respect to N and to the number of shifts, and it is halved with respect to straightforward application of structured QR factorization to each shifted linear system. The Matlab code is available at `http://www.unilim.fr/pages_perso/paola.boito/software.html`.

Some applications are presented in the following sections; they include the computation of $f(A)\boldsymbol{v}$ via series expansion or contour integrals, and the solution of linear matrix equations.

3 A Model Problem for Boundary Value ODEs

Consider the non-local boundary value problem:

$$\frac{d\boldsymbol{v}}{dt} = A\boldsymbol{v}, \quad 0 < t < \tau,$$

$$\frac{1}{\tau}\int_0^\tau \boldsymbol{v}(t)\, dt = \boldsymbol{g},$$

where A is a linear operator in $\mathbb{R}^N$ and $\boldsymbol{g} \in \mathbb{R}^N$ is a given vector [11, 12, 5].

If all the numbers $\mu_k = 2\pi i k/\tau,\ k = \pm 1, \pm 2, \pm 3, \dots$ are regular points of the operator A, the problem has a unique solution, given by

$$\boldsymbol{v}(t) = q_t(A)\boldsymbol{g}, \quad q_t(z) = \frac{\tau z e^{zt}}{e^{z\tau} - 1}.$$

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Without loss of generality one can assume $\tau = 2\pi$.

We show that, for a suitable truncation index ℓ, $\boldsymbol{v}(t)$ can be approximated by the finite sum

$$\boldsymbol{v}_\ell(t) = \sum_{j=0}^{3} V_j(t) A^j g - 2 \sum_{k=1}^{\ell} \frac{1}{k^2}(A\cos kt + \frac{1}{k}A^2 \sin kt)(A^2 + k^2 I_N)^{-1} A^3 g,$$

with $0 \le t \le 2\pi$ and

$$V_0(t) = 1,\ V_1(t) = t - \pi,\ V_2(t) = \frac{\pi^2}{3} - \pi t + \frac{t^2}{2},\ V_3(t) = \frac{\pi^2}{3}t - \frac{\pi}{2}t^2 + \frac{t^3}{6}.$$

The computation of $\boldsymbol{v}(t_i)$, $0 \le i \le M+1$, requires the solution of a possibly large set of shifted systems and our algorithm proves to be effective for this task if A is quasiseparable.

More generally, a similar approach can be applied to the computation of a function of a quasiseparable matrix, multiplied by a vector, whenever the function can be represented as a series of partial fractions. The classes of meromorphic functions admitting such a representation were investigated for instance in [8]. Other partial fraction approximations of certain analytic functions can be found in [7].

4 Sylvester-type Matrix Equations

As a natural extension of the problem (1), the right-hand side $\boldsymbol{y}$ could also depend on the parameter, so that we have a different right-hand side for each linear system. This situation is common in many applications, such as control theory, structural dynamics and time-dependent PDEs [6]. The systems to be solved take the form of a linear matrix equation:

$$AX + XD = Y, \quad A \in \mathbb{R}^{N\times N},\ D = \mathrm{diag}\left[\sigma_1, \ldots, \sigma_\ell\right],\ Y = \left[\boldsymbol{y}_1, \ldots, \boldsymbol{y}_\ell\right].$$

The extension to the case where D is lower triangular can be carried out via backsubstitution, and a further generalization to the case where D is a general matrix is possible using the classical Bartels-Stewart approach based on Schur decomposition. This approach is especially interesting when ℓ is significantly smaller than N. If A is quasiseparable we can apply our structured approach as outlined above, with computational advantages with respect to the widespread method that relies on Kronecker products.

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